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On Convergence of Iterates of Positive Contractions in L_p Spaces*

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1. INTRODUCTION

We consider in this paper relations between weak and strong convergence of iterates of positive contractions in L_p spaces, $1 < p < \infty$. A matrix of real numbers $(a_{ni})_{n,i=1,2,\dots}$ is called uniformly regular if the following conditions hold:

$$\sup_n \sum_i | a_{ni} | < \infty; \qquad \lim_n \max_i | a_{ni} | = 0; \qquad \lim_n \sum_i a_{ni} = 1. \qquad (1.1)$$

G. G. Lorentz characterized the class of uniformly regular methods in terms of "summability functions" (see [9 and 10]). Here we study the problem of equivalence of the following two conditions (A) and (B):

(A) T^n converges weakly in L_p ;

(B) $\sum_i a_{ni} T^i$ converges strongly in L_p for every uniformly regular matrix (a_{ni}).

The implication (B) $\Rightarrow$ (A) is easy, and in fact, as observed in [7], (A) is implied in arbitrary Banach spaces by a condition in appearance weaker than (B), namely the existence of a regular matrix (a_{ni}) such that $\sum_i a_{ni} T^{k_i}$ converges weakly for every strictly increasing sequence of positive integers (k_i). In Section 2 of the present paper we show that (A) implies (B) if T is

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a positive invertible isometry and the weak limit $\bar{T}$ of T^n is zero. Section 3 proves the same result in the case when $\bar{T}$ is not zero. In Section 5 an attempt is made to extend the implication (A) $\Rightarrow$ (B) to positive *contractions*. An additional condition is needed: that there exists a function h in L_p such that $h > 0$ a.e. and $\| Th \| = \| h \|$. Under the same condition it is at first shown in Sections 4 and 5 that a contraction T on L_p has a dilation to an isometry, and this isometry generates the decomposition of the space X into two invariant parts, X_1 and X_2 : On X_1 the theorem holds because the isometry is invertible; on X_2 because this part is a disjoint union of images of a "wandering" set.

The question whether (A) implies (B) for general positive contractions remains open.[1] For $p = 1$ and 2, however, the answer is yes, and the implication (A) $\Rightarrow$ (B) holds even if T is not positive (cf. [5 and 7]).

2. Convergence of Invertible Isometries to Zero

Let (X, Σ, m) be a σ-finite measure space and let $L_p = L_p(X, \Sigma, m)$ be the usual Banach spaces. In the proofs, we assume without loss of generality that $m(X) = 1$: L_p of a σ-finite measure space is isometric and lattice isomorphic to L_p of a probability space. If $E \in \Sigma$, then $L_p(E, m)$ or $L_p(E)$ denotes the subspace of L_p consisting of functions with support in E. A *contraction* T on L_p is a linear operator on L_p of norm $\leqslant 1$. L_p^+ is the cone of nonnegative elements of L_p. Since L_p is actually composed of *equivalence classes* of functions, many statements below are to be understood modulo sets of m-measure zero, or modulo m-null functions.

The inner product $\int f \cdot g \, dm$ is denoted (f, g). If p is a number between 1 and ∞, p' denotes $p/(p-1)$.

LEMMA 2.1. *Assume $p > 1$. Let $f \in L_p^+$. The unique element of $L_{p'}^+$ satisfying the equation in g*

$$(g, f) = \|f\|_p \, \|g\|_{p'} = \|f\|_p^p,$$

is the function $g = f^{p-1}$.

Proof. f^{p-1} satisfies the equation. The equality in Hölder's inequality $(f, g) \leqslant \|f\|_p \, \|g\|_{p'}$ determines g up to a multiplicative constant, which must be 1 because of the second equality in the equation.

[1] Added in proof: The answer is yes. (See a research announcement by the present authors in *Bull. Amer. Math. Soc.*, January 1975.)

LEMMA 2.2. *If T is an invertible isometry on L_p, $1 < p < \infty$, then T^* is an isometry on $L_{p'}$.*

Proof. Given $h \in L_{p'}$ with $\| h \| = 1$, choose $g \in L_p$ so that $\| h \| = (h, g) = \| g \|$. Let f be such that $Tf = g$. Then $(h, Tf) = \| Tf \| = \| f \|$. Also, $(h, Tf) = (T^*h, f) \leqslant \| T^*h \| \| f \|$. Thus $\| T^*h \| \geqslant 1$, hence $\| T^*h \| = 1 = \| h \|$.

LEMMA 2.3. *If T is an invertible isometry on L_p, then for each $f \in L_p^+$, $T^*f^{p-1} = (T^{-1}f)^{p-1}$.*

Proof. By Lemma 2.1, $(T^{-1}f)^{p-1}$ is the unique element of $L_{p'}$ satisfying the equation in g

$$(T^{-1}f, g) = \| T^{-1}f \| \| g \| = \| f \| \| g \| = \| f \|^p.$$

Because of Lemma 2.2, this equation is also satisfied by $g = T^*f^{p-1}$. Hence $(T^{-1}f)^{p-1} = T^*f^{p-1}$.

LEMMA 2.4. *Let $1 \leqslant p_1 \leqslant q \leqslant p_2 \leqslant \infty$. Then for each $f \in L_{p_2}$ one has*

$$\| f \|_q^q \leqslant (\| f \|_{p_1})^\alpha (\| f \|_{p_2})^{q-\alpha},$$

where $\alpha = p_1(p_2 - q)/(p_2 - p_1)$.

Proof. Assume at first that $p_2 < \infty$. Set $\beta = p_2 \times (q - p_1)/(p_2 - p_1)$, $t = (p_2 - p_1)/(p_2 - q)$, $t' = t/(t - 1)$. Then $\alpha + \beta = q$, $\alpha t = p_1$, $\beta t' = p_2$. Hölder's inequality implies that

$$\begin{aligned} \| f \|_q^q &= (| f |^\alpha, | f |^\beta) \leqslant \| f^\alpha \|_t \cdot \| f^\beta \|_{t'} \\ &= (\| f \|_{p_1})^\alpha \cdot (\| f \|_{p_2})^\beta. \end{aligned} \tag{2.5}$$

The case $p_2 = \infty$ is obtained by passing to the limit in the inequality of the lemma (cf., e.g., Loève [8, p. 160]).

Since the limit in the following theorem is zero, a condition on the matrix weaker than uniform regularity is sufficient, namely

$$\sup_n \sum_i | a_{ni} | < \infty \qquad \text{and} \qquad \lim_n \max_i | a_{ni} | = 0. \tag{2.6}$$

THEOREM 2.7. *Let T be a positive invertible isometry on L_p where p is a fixed number, $1 < p < \infty$. The following conditions (α) and (β) are equivalent.*

(α) T^n *converges weakly to zero*; (β) $\sum_i a_{ni}T^i$ *converges strongly to zero for each matrix* (a_{ni}) *satisfying* (2.6).

Proof. As observed in the introduction, it suffices to prove that (α) implies (β). Assume (α). Write S_n for the operator $\sum_i a_{ni}T^i$. We prove that $\| S_n f \|_p^p = ((S_n f)^{p-1}, S_n f) \to 0$. By Lemma 2.3, $T^{*j}(S_n f)^{p-1} = (T^{-j}S_n f)^{p-1} = (S_n T^{-j} f)^{p-1}$. Hence it suffices to show that for each $f \in L_p^+$

$$\left(\sum_j a_{nj} \left(\sum_i a_{ni} T^{i-j} f\right)^{p-1}, f\right) \to 0. \tag{2.8}$$

Since L_∞ is a dense subspace of L_p and T is bounded, we may, and do, assume that the function f is bounded by 1. Instead of (2.8) we will prove the stronger statement

$$\sum_j a_{nj} \int \left(\sum_i a_{ni} T^{i-j} f\right)^{p-1} dm \to 0. \tag{2.9}$$

We require the following lemma.

LEMMA 2.10. *Let* (d_{ij}) *be a matrix of real numbers bounded by* 1 *and such that* $\lim_{|i-j|\to\infty} d_{ij} = 0$. *Let* (a_{ni}) *satisfy* (2.6). *Then for each real number* $q > 0$

$$\sum_j | a_{nj} | \left(\sum_i | a_{ni} | d_{ij}\right)^q \to 0.$$

Proof. Set $\sup_n \sum_i | a_{ni} | = m$. Let $\epsilon > 0$ be given and choose K so large that $| i - j | > K$ implies $| d_{ij} | < \epsilon$. Select N so large that for $n > N$ one has $\max_i | a_{ni} | < \epsilon$. For each positive integer j there are at most $2K + 1$ terms d_{ij} such that $| d_{ij} | < \epsilon$ need not hold. Thus for $n > N$ we can write, using the convention that $a_{ni} = 0$ for $i \leqslant 0$,

$$\sum_i | a_{ni} | d_{ij} \leqslant \sum_{i=j-K}^{j+K} | a_{ni} | \cdot 1 + \epsilon \sum_i | a_{ni} | \leqslant (2K + 1 + m) \cdot \epsilon,$$

hence

$$\sum_j | a_{nj} | \left(\sum_i | a_{ni} | d_{ij}\right)^q \leqslant m(2K + 1 + m)^q \epsilon^q,$$

which proves the lemma, since ϵ is arbitrary.

We continue the proof of the theorem.

Case $p < 2$. One has for each $g \in L_+^{p-1}$ (cf. Loève [8, p. 156])

$$\left(\int g^{p-1}\, dm\right)^{1/(p-1)} \leqslant \int g\, dm,$$

hence the expression in (2.9) is bounded by

$$\sum_j |a_{nj}| \left[\sum_i |a_{ni}| \left(\int T^{i-j} f\, dm\right)\right]^{p-1}.$$

(2.9) now follows from Lemma 2.10 applied with $d_{ij} = \int T^{i-j} f\, dm$ and $q = p - 1$, because weak convergence to zero of $T^n f$ in L_p implies that $\int T^n f\, dm \to 0$.

Case $p \geqslant 2$. Minkowski's inequality is now available in L_{p-1}, which implies that instead of (2.9) it suffices to prove

$$\sum_j |a_{nj}| \left[\sum_i |a_{ni}| \| T^{i-j} f \|_{p-1}\right]^{p-1} \to 0. \tag{2.11}$$

Observe that $\| T^{-n} f \|_{p-1} \to 0$ because $(T^{-n} f)^{p-1} = T^{*n} f^{p-1}$ and weak convergence to zero of T^n in L_p implies weak convergence to zero of T^{*n} in $L_{p'}$. Also $\| T^n f \|_{p-1}$ converges to zero, because Lemma 2.4 may be applied with $p_1 = 1$, $p_2 = p$, $q = p - 1$. This proves that

$$\lim_{|i-j| \to \infty} \| T^{i-j} f \|_{p-1} = 0. \tag{2.12}$$

(2.11) will now be a consequence of the Lemma 2.10 applied with $d_{ij} = \| T^{i-j} f \|_{p-1}$ and $q = p - 1$.

3. Convergence of Isometries to a Positive Limit

We now require a decomposition of the space X such that on one part of the space the weak limit is zero, and on the other part there is a positive fixed point.

Proposition 3.1. *Let T be a positive, linear operator on L_p, $1 \leqslant p < \infty$. Then X uniquely decomposes into two sets F and G with the following properties. G is the support of a T-invariant nonnegative function g_0, and the support of any such function is contained in G. G is invariant; i.e., $f \in L_p(G)$ implies $Tf \in L_p(G)$. If $f \in L_p^+$ and $T^n f$ converges weakly, then $\int_F T^n f\, dm \to 0$.*

Proof. Let $Q = \{g: g \in L_p^+, Tg = g\}$, $G = \bigcup_{g \in Q} \operatorname{supp} g$. Define a function $g_0 \in L_p^+$ as follows. If $G = \varnothing$, set $g_0 = 0$. Otherwise there is a countable sequence of nonnull functions g_i in Q so that $G = \bigcup_i \operatorname{supp} g_i$. Set $g_0 = \sum_i \alpha_i g_i$ where α_i are positive constants so chosen that $g_0 \in L_p$. Then $G = \operatorname{supp} g_0$ and $g_0 \in Q$. The set G is invariant, because, given $f \in L_p^+$ and an $\epsilon > 0$, we can set $f = f_1 + f_2$, where $\|f_1\| < \epsilon$, and there is a positive constant c such that $f_2 < cg_0$, hence $cg_0 \geqslant Tf_2 \in L_p^+$. Define an operator R on $L_p(F)$ by $Rf = 1_F(Tf)$, $f \in L_p(F)$. Clearly $R^n(1_F f) = 1_F T^n f$ for all n. If weak-lim $T^n f = \bar{f}$, then $T\bar{f} = \bar{f}$, hence supp $f \subset G$, and $\lim_n \int_F T^n f\, dm = \int_F \bar{f}\, dm = 0$. □

COROLLARY 3.2. *If $p > 1$ and T satisfies*

$$\sup_n \left\| \frac{1}{n} \sum_1^n T^i \right\| < \infty \qquad \textit{and} \qquad \frac{T^n}{n} \to 0 \quad \textit{strongly},$$

then for each $f \in L_p$

$$\int_F \left(\frac{1}{n} \sum_1^n T^i f \right)^p dm \to 0.$$

Proof. The mean ergodic theorem applied to R in $L_p(F)$ gives that the Cesàro averages of $R^n f$ converge to a function $\bar{f} \in L_p(F)$. $R\bar{f} = \bar{f}$, hence $T\bar{f} = \bar{f}$, supp $\bar{f} \subset G$ and $\bar{f} = 0$.

A bounded linear operator T is called invertible if it is one-to-one and onto. The inverse T^{-1} is then a bounded linear operator by a well known theorem of Banach, and T^{-1} is positive if T is positive. In some cases the decomposition $X = F + G$ is more satisfying in that not only G, but also F is invariant, so that the separation between the two sets is complete.

COROLLARY 3.3. *If T is invertible, then both F and G are invariant.*

Proof. T and T^{-1} have the same decomposition $F + G$, because they have the same fixed points, since $Tg = g$ implies $g = T^{-1}(Tg) = T^{-1}g$. Let $Tf = f_1 + g_1$ with $f_1 \in L_p(F)$, $g_1 \in L_p(G)$. Then $T^{-1}g_1 \in L_p(G)$, hence $f = T^{-1}(f_1 + g_1) = T^{-1}f_1$, $Tf = f_1$ and $g_1 = 0$.

PROPOSITION 3.4. *If T is a positive contraction on L_p, $1 < p < \infty$, and $Tf = f \in L_p^+$, then both $E = \operatorname{supp} f$ and E^c are invariant.*

Proof. Assume $\|f\|_p = 1$. Then $1 = (f, f^{p-1}) = (Tf, f^{p-1}) = (f, T^* f^{p-1}) \leqslant \| T^* f^{p-1} \|$. On the other hand $\| T^* f^{p-1} \| \leqslant \| T^* \| \| f^{p-1} \| \leqslant 1$. Thus $\| T^* f^{p-1} \| = 1 = \| f^{p-1} \|$. This implies that $T^* f^{p-1}$ is a solution of the equation in g appearing in Lemma 2.1. Hence $T^* f^{p-1} = f^{p-1}$. The set E is the support of a T-invariant function f, hence is T-invariant (cf. the proof of invariance of G in 3.1), and E is also the support of a T^*-invariant function f^{p-1}, hence E is T^*-invariant. It follows that E^c is T invariant. To see this, let $f_1 \in L_p(E^c)$, $f_2 \in L_{p'}(E)$, and note that $(Tf_1, f_2) = 0$ if and only if $(f_1, T^*f_2) = 0$.

COROLLARY 3.5. *If T is a positive contraction on L_p, $1 < p < \infty$, then the sets F and G appearing in Proposition* 3.1 *are both invariant.*

Either 3.3 or 3.5 may be used to show that the problem of equivalence of (A) and (B) for invertible isometries may be studied separately on the parts F and G of the space. Section 2 resolves this problem for the part F. For the part G, the following theorem proved in ([7, Section 2]) is applicable:

THEOREM 3.6. *Let T be a positive contraction on L_p, $1 < p < \infty$. If there is a function $h \in L_p$ such that $h > 0$ a.e. and $Th = h$, then the conditions* (A) *and* (B) *are equivalent.*

We only very briefly sketch the proof; for details see [7]. Define a measure γ on Σ by $d\gamma = h^p\, dm$. Define an operator S on $L_p(X, \gamma)$ by $hSf = T(hf)$, then $h^{p-1}S^*f = T^*(h^{p-1}f)$. One verifies that S is a contraction on $L_p(X, \gamma)$, and both S and S^* are contractions on $L_\infty(X, \gamma)$, hence S is also a contraction on $L_1(X, \gamma)$. It follows that S is a contraction on $L_2(X, \gamma)$, and on L_2 the equivalence of (A) and (B) is rather easy to prove. From the validity of the equivalence for S one derives the validity for T, hence the theorem.

(2.7), (3.5) and (3.6) now imply:

THEOREM 3.7. *The conditions* (A) *and* (B) *are equivalent for arbitrary invertible isometries on L_p, $1 < p < \infty$.*

4. Dilations of Contractions in L_p-Spaces

In this section we will prove Theorem 4.1 below, which will be used in the next section. Similar theorems were obtained in [2] and [3], for the finite dimensional L_p-Spaces and for the L_1-Spaces, respectively.

Before we state this main result, we recall the following definitions and theorems. An equivalence between two measurable spaces is an invertible point transformation which is measurable in both directions. An isomorphism between two measure spaces is a measure preserving point transformation that becomes an equivalence if a null set is omitted from each one of the spaces. A Borel space is any measure space that is isomorphic to (J, β, μ), where $J = [0, 1]$ is the unit interval, β is the σ-algebra of its Borel sets and μ

is a finite measure. If $\phi: J \to J$ is an isomorphism between (J, β, μ) and (J, β, ν) then there is also a measure preserving equivalence $\psi: J \to J$ between these measure spaces. In what follows (J_i, β_i), $i = 0, \pm 1, \pm 2, \ldots$, will denote copies of (J, β) and we will let $(J_k^l, \beta_k^l) = \prod_{i=k}^{l} (J_i, \beta_i)$, $-\infty \leqslant k \leqslant l \leqslant \infty$. Note that (J_k^l, β_k^l) is always equivalent to (J, β).

A nonsingular equivalence τ of a measure space (X, Σ, m) is an equivalence that transports m to an measure $m\tau^{-1}$ absolutely continuous with respect to m. A nonsingular equivalence τ of (X, Σ, m) induces a positive isometry Q of $L_p(X, \Sigma, m)$, defined as

$$(Qf)(x) = \left[\frac{dm\tau^{-1}}{dm}(x)\right]^{1/p} f(\tau^{-1}x).$$

THEOREM 4.1. *Let T be a positive contraction of $L_p(J, \beta, \mu)$, where μ is a normalized measure and $1 < p < \infty$. Assume that there is a function $h \in L_p(J, \beta, \mu)$ so that $h > 0$ a.e. and $\| Th \| = \| h \|$. Then there exists another normalized measure $\tilde{\mu}$ on β and a non singular equivalence τ of $(J, \beta, \tilde{\mu})$ so that*

(i) *There is a sub σ-algebra $\mathscr{C} \subset \beta$ and a positivity preserving isomorphism $\xi: L_p(J, \beta, \mu) \to L_p(J, \mathscr{C}, \tilde{\mu})$,*

(ii) *If Q is the positive isometry of $L_p(J, \beta, \tilde{\mu})$ induced by τ and if*

$$E: L_p(J, \beta, \tilde{\mu}) \to L_p(J, \mathscr{C}, \tilde{\mu})$$

is the conditional expectation operator with respect to $\mathscr{C}$, then $\xi T^n f = EQ^n \xi f$ for each $f \in L_p(J, \beta, \mu)$ and for each $n = 0, 1, 2, \ldots$.

The proof will depend on several lemmas. In this proof the measure $\tilde{\mu}$ will actually be constructed as a measure $\mu_{-\infty}^{\infty}$ on the cartesian product space $(J_{-\infty}^{\infty}, \beta_{-\infty}^{\infty})$. Similarly, τ will be a nonsingular equivalence of $(J_{-\infty}^{\infty}, \beta_{-\infty}^{\infty}, \mu_{-\infty}^{\infty})$ and $\mathscr{C} = \beta_0 \subset \beta_{-\infty}^{\infty}$ will be the sub σ-algebra of $\beta_{-\infty}^{\infty}$ generated by the J_0-coordinate function $J_{-\infty}^{\infty} \to J_0$. Since $(J_{-\infty}^{\infty}, \beta_{-\infty}^{\infty})$ is equivalent to (J, β), the formulation given in the theorem may then be obtained easily.

The measure $\mu_{-\infty}^{\infty}$ will be constructed in such a way that its projection on the coordinate space (J_0, β_0) will be μ. Hence the isomorphism ξ will amount to identifying a function on J as a function on $J_{-\infty}^{\infty}$ that depends only on the J_0-coordinate.

DEFINITION 4.2. Let $(X, \mathscr{F})$ and $(Y, \mathscr{G})$ be two measurable spaces and let $\{\eta\} = \{\eta\}_X$ be a family of normalized measures on $(Y, \mathscr{G})$, indexed by the elements of X. Then $\{\eta\}$ is called a conditioned family if the values of these measures at each $G \in \mathscr{G}$ define a measurable function on $(X, \mathscr{F})$. If σ is a measure on $(X, \mathscr{F})$ and $\{\eta\}_X$ is a conditioned family of measures on $(Y, \mathscr{G})$

then $\sigma \times \{\eta\}$ will denote the measure on $(X, \mathscr{F}) \times (Y, \mathscr{G})$ defined uniquely by the condition that

$$(\sigma \times \{\eta\})(F \times G) = \int_F \eta(G, x)\, \mu(dx),$$

for each $F \in \mathscr{F}$ and $G \in \mathscr{G}$. Here $\eta(\cdot, x)\colon \mathscr{G} \to [0, 1]$ is the member of $\{\eta\}$ corresponding to $x \in X$.

LEMMA 4.3. *There exist a conditioned family $\{\alpha\}_{J_0}$ on (J_{-1}, β_{-1}) and an equivalence $\pi\colon (J^0_{-1}, \beta^0_{-1}) \to (J_0{}^1, \beta_0{}^1)$ so that π transports $\{\alpha\} \times \mu$ to $\nu \times \lambda$, where $d\mu = h^p\, dm$, $d\nu = (Th)^p\, dm$ and where λ is the standard Lebesgue measure, so that*

$$(Tf)(x_0) = (Th)(x_0) \int_{J_1} \frac{f(\pi_0^{-1}(x_0, x_1))}{h(\pi_0^{-1}(x_0, x_1))}\, dx_1,$$

for each $f \in L_p(J, \beta, m)$. Here $\pi_0^{-1}(x_0, x_1)$ denotes the J_0-coordinate of the point $\pi^{-1}(x_0, x_1) \in J^0_{-1}$.

Proof. Let $S_n = \{0, 1\}^n$ be the set of all sequences of length n of zeros and ones and let $S = \bigcup_{n=1}^{\infty} S_n$ be the set of all such finite sequences. For each $s \in S$, let j_s be the corresponding binary interval of $J = [0, 1]$, consisting of the numbers whose binary expansions start with the sequence s. We will assume that the end points of these intervals are so adjusted that for each $n \geqslant 1$ the family $\{j_s\}_{s \in S_n}$ is a partition of J and that these partitions get finer as n increases. Let X_s be the characteristic function of j_s.

For each $s \in S$, we are now going to define a subset G_s of $J_0{}^1$ so that the following conditions will be satisfied

4.4. For each $n \geqslant 1$, $\{G_s\}_{s \in S_n}$ is a partition of $J_0{}^1$ and these partitions get finer as n increases.

4.5. If $s, s' \in S'$ and if s' is an extension of s then $G_{s'} \subset G_s$.

4.6. If ψ_s is the characteristic function of G_s then

$$T(X_s h)(x_0) = (Th)(x_0) \int_{J_1} \psi_s(x_0, x_1)\, dx_1,$$

m — a.a. $x_0 \in J_0$.

To define G_s's, let $p_s(x_0) = (T(X_s h)(x_0))/(Th(x_0))$. We may assume that these functions are so adjusted that the following countably many conditions are satisfied at each point.

$$p_0 + p_1 = 1,$$

$$p_{s0} + p_{s1} = p_s, \qquad s \in S,$$

where $s0$ and $s1$ denote the extensions of s by a 0 or by a 1, respectively. We may then let, for example,

$$G_0 = \{(x_0, x_1) \mid 0 \leqslant x_0 \leqslant 1, 0 \leqslant x_1 < p_0(x_0)\},$$
$$G_1 = \{(x_0, x_1) \mid 0 \leqslant x_0 \leqslant 1, p_0(x_0) \leqslant x_1 \leqslant 1\}$$

and continue the definition of G_s's by an obvious induction.

These sets define a function $g: J_0{}^1 \to J$ as follows. For each $(x_0, x_1) \in J_0{}^1$, $g(x_0, x_1)$ is the real number whose binary expansion is given by the indices of G_s's that contain this point. Then g transports $\nu \times \lambda$ to μ. In fact, for each binary interval $j_s \subset J$,

$$(\nu \times \lambda)(g^{-1}j_s) = (\nu \times \lambda)(G_s) = \int_{J_0} \frac{TX_s h}{Th} (Th)^p \, dm$$
$$= \int_{J_0} (TX_s h)(Th)^{p-1} \, dm = \int_{J_0} X_s h T^*(Th)^{p-1} \, dm$$
$$= \int_{J_0} X_s h h^{p-1} \, dm = \int_{j_s} h^p \, dm = \mu(j_s).$$

Therefore, by Rohlin's theorem [12, 1], there exists a conditioned family $\{\alpha\}_{J_0}$ on (J_{-1}, β_{-1}) and an equivalence

$$\pi: J_{-1}^0 \to J_0{}^1,$$

so that π transports $\{\alpha\} \times \mu$ to $\nu \times \lambda$ and so that $g\pi: J_{-1}^0 \to J_0$ is the projection of J_{-1}^0 to its J_0-component, $\{\alpha\} \times \mu$ — a.e.

To see that

$$(Tf)(x_0) = (Th)(x_0) \int_{J_0} \frac{f(\pi_0^{-1}(x_0, x_1))}{h(\pi_0^{-1}(x_0, x_1))} \, dx_1$$

for each $f \in L_p(J, \beta, m)$, we observe that this equation is true if $f = X_s h$, by the definition of π. Hence it is also true for any $f = \phi h$, with $\phi \in L_p(J, \beta, \mu)$, which includes all $f \in L_p(J, \beta, m)$.

4.7. We will now construct an equivalence $\tau: J_{-\infty}^{\infty} \to J_{-\infty}^{\infty}$ as follows. If x_i and $\tau_i{}^n x$ denote the ith coordinates of $x \in J_{-\infty}^{\infty}$ and $\tau^n x \in J_{-\infty}^{\infty}$, respectively, then

$$\tau_0 x = \pi_0(x_{-1}, x_0),$$
$$\tau_1 x = \pi_1(x_{-1}, x_0),$$
$$\tau_i x = x_{i-1} \qquad \text{if} \quad i \neq 0, \quad i \neq 1,$$

where $(x_{-1}, x_0) \to (\pi_0(x_{-1}, x_0), \pi_1(x_{-1}, x_0))$ denotes the equivalence $\pi: J_{-1}^0 \to J_0^1$ as constructed above. We will define two measures (see [3] for further details)

$$\mu_{-\infty}^{\infty} = \times\{\alpha_{-2}\} \times \{\alpha_{-1}\} \times \mu \times \lambda \times \lambda \times \cdots$$

and

$$\nu_{-\infty}^{\infty} = \cdots \times \{\alpha_{-2}\} \times \{\alpha_{-1}\} \times \nu \times \lambda \times \lambda \times \cdots$$

where $d\mu = h^p\, dm$, $d\nu = (Th)^p\, dm$ are measures on (J_0, β_0), as before, and λ is the standard Lebesgue measure, and for each $n = 1, 2,\ldots, \{\alpha_{-n}\}_{J_0^{-n+1}}$ is a conditioned family on (J_{-n}, β_{-n}) defined as follows:

$$\alpha_{-n}(F, (x_{-n+1}, \ldots, x_0)) = \alpha(F, \tau_0^{n-1}(x_{-n+1}, \ldots, x_0)),$$

where $\{\alpha\}$ is the conditioned family obtained previously.

Then one can check that τ transports $\mu_{-\infty}^{\infty}$ to $\nu_{-\infty}^{\infty}$ and also that

$$\frac{d\nu_{-\infty}^{\infty}}{d\mu_{-\infty}^{\infty}}(\ldots, x_{-1}, x_0, x_1, \ldots) = \frac{d\nu}{d\mu}(x_0) = \frac{(Th)^p\,(x_0)}{h^p(x_0)}.$$

If Q is the positive isometry induced on $L_p(J_{-\infty}^{\infty}, \beta_{-\infty}^{\infty}, \mu_{-\infty}^{\infty})$ by τ, then

$$(QF)(x) = \frac{(Th)(x_0)}{h(x_0)} F(\tau^{-1}x),$$

for each $F \in L_p(J_{-\infty}^{\infty}, \beta_{-\infty}^{\infty}, \mu_{-\infty}^{\infty})$ and $x \in J_{-\infty}^{\infty}$, $x_0 \in J_0$ being the J_0-coordinate of x.

Each $\phi \in L_p(J, \beta, \mu)$ can also be considered as a member of $L_p(J_{-\infty}^{\infty}, \beta_{-\infty}^{\infty}, \mu_{-\infty}^{\infty})$, depending only on the J_0-coordinate x_0 of a point $x \in J_{-\infty}^{\infty}$. Then $\phi h \in L_p(J, \beta, m)$, and we would like to show that

$$T^n(\phi h) = hEQ^n\phi,$$

for each $n = 0, 1, 2,\ldots$, and for each $\phi \in L_p(J, \beta, \mu)$, or equivalently, that

$$T^n f = hEQ^n(f/h), \qquad \text{for each} \quad n = 0, 1, 2, \ldots$$

and for each $f \in L_p(J, \beta, m)$, where $E: L_p(J_{-\infty}^{\infty}, \beta_{-\infty}^{\infty}, \mu_{-\infty}^{\infty}) \to L_p(J, \beta, \mu)$ is the conditional expectation operator with respect to $\beta_0 \subset \beta_{-\infty}^{\infty}$, the σ-algebra generated by the J_0-coordinates.

The proof is by induction and essentially depends on the following lemma.

LEMMA 4.8. *Let $F \in L_p(J_{-\infty}^{\infty}, \beta_{-\infty}^{\infty}, \mu_{-\infty}^{\infty})$ be a function depending only on finitely many coordinates $(x_0, x_1, \ldots, x_n)$, $n \geq 0$. Then $EQF = EQEF = (1/h)\, ThEF$.*

Proof. Write the value of F at $(..., x_{-1}, x_0, x_1, ...)$ as $F(x_0, ..., x_n)$. Observing that QF depends only on $(x_0, ..., x_{n+1})$, we may write

$$(QF)(x_0, ..., x_{n+1}) = \frac{(Th)(x_0)}{h(x_0)} F(\pi_0^{-1}(x_0, x_1), x_2, ..., x_{n+1}).$$

Hence

$$\begin{aligned}(EQF)(x_0) &= \frac{(Th)(x_0)}{h(x_0)} \int_{J_1^{n+1}} F(\pi_0^{-1}(x_0, x_1), x_2, ..., x_{n+1})\, dx_1 \cdots dx_{n+1} \\ &= \frac{(Th)(x_0)}{h(x_0)} \int_{J_1} dx_1 \int_{J_2^{n+1}} F(\pi_0^{-1}(x_0, x_1), x_2, ..., x_{n+1})\, dx_2 \cdots dx_{n+1} \\ &= \frac{(Th)(x_0)}{h(x_0)} \int_{J_1} (EF)(\pi_0^{-1}(x_0, x_1))\, dx_1 \\ &= \frac{1}{h(x_0)} (T(hEF))(x_0), \qquad \text{by Lemma 3.}\end{aligned}$$

Similarly, $EQEF = (1/h)\, T(hEF)$.

Now to prove the main equation, namely

$$T^n f = hEQ^n(f/h),$$

for each $n = 0, 1, ...,$ and for each $f \in L_p(J, \beta, m)$ observe that this equation is trivial for $n = 0$. If it is true for n, since $Q^n(f/h)$ depends only on $(x_0, ..., x_n)$, then we have that

$$\begin{aligned}EQ^{n+1} \frac{f}{h} &= EQQ^n \frac{f}{h} = EQEQ^n \frac{f}{h} \\ &= EQ \frac{1}{h} T^n f = \frac{1}{h} T^{n+1} f \quad \square\end{aligned}$$

5. Positive Contractions

In this section we prove the final result of this note.

THEOREM 5.1. *Let T be a positive contraction on $L_p = L_p(X, \Sigma, m)$, $1 < p < \infty$, and assume that there is a function $h \in L_p^+$ so that $h > 0$ a.e. and $\| Th \| = \| h \|$. T^n converges weakly (if and) only if $\sum_i a_{ni} T^i$ converges strongly for every uniformly regular matrix (a_{ni}). Then one has weak-$\lim T^n = \lim \sum_i a_{ni} T^i$.*

The proof will consist of several separate arguments. As shown in Section 3, the space X can be decomposed into two invariant sets G and

$F = X - G$, so that any invariant function of T has support in G and there is an invariant function strictly positive a.e. on G. Then the restrictions of T to $L_p(G, m)$ and to $L_p(F, m)$ can be considered separately. For the first part, the results of Section 3 apply and we obtain the proof immediately. For the second part, note that $h1_F$ is also a function satisfying $\| Th1_F \| = \| h1_F \|$. We will also observe that there is no loss of generality in replacing the measure space by a Borel space. Hence, because of the dilation theorem of the previous section, T has a dilation to an isometry Q. Since both T and Q are positive and since T^n converges weakly to zero, it is clear that Q^n satisfying

$$\xi T^n = EQ^n\xi \tag{5.2}$$

also converges weakly to zero. Hence we need the following theorem to obtain the desired result.

THEOREM 5.3. *Let Q be a positive isometry of $L_p(X, \Sigma, m)$, induced by a nonsingular equivalence τ of X. Then the weak convergence of Q^n to zero implies the strong convergence of $\sum_i a_{ni}Q^i f$ for any matrix (a_{ni}) satisfying* (2.6).

Proof. For each $n = 0, 1, 2,...$ let $m\tau^{-n}$ be the measure transported by τ^n, and let X_n be a set with the minimal m-measure so that $m\tau^{-n}X_n = m\tau^{-n}X (= mX)$. We may and will assume that $X_0 \supset X_1 \supset X_2 \supset \cdots$. Let $D_n = X_n - X_{n-1}$, $n = 1, 2,...$, and $A = \bigcup_{n=1}^{\infty} D_n$, $B = \bigcap_{n=1}^{\infty} X_n = X - A$. Then it is easy to see that if C is a subset of D_n with $m(C) > 0$, then τC is essentially a subset of D_{n+1} with $m(\tau C) > 0$, $n = 1, 2,...$, and $\tau^{-1}C$ is essentially a subset of D_{n-1} with $m(\tau^{-1}C) > 0$, $n = 2, 3,...$. Similarly if C is a nonzero subset of B then both τC and $\tau^{-1}C$ are essentially nonzero subsets of B. Hence Q maps $L_p(D_n, m)$ onto $L_p(D_{n+1}, m)$, and it is also an invertible isometry of $L_p(B, m)$ onto itself.

Now if $f \in L_p(B, m)$, then $\sum_i a_{ni}Q^i f$ converges strongly to zero, by the results of Section 2. If $f \in L_p(D_n, m)$ for some $n = 1, 2,...$, then again $\sum_i a_{ni}Q^i f$ converges strongly to zero for the following reason. First, since $Q^i f$'s have disjoint supports,

$$\Big\| \sum_i a_{ni}Q^i f \Big\|_p^p = \sum_i | a_{ni} |^p \| Q^i f \|_p^p = \| f \|_p^p \sum_i | a_{ni} |^p.$$

Hence it is enough to show that $\lim_{n\to\infty} \sum_i a_{ni}^p = 0$. In fact, if $m = \sup_n \sum_i | a_{ni} |$ and if $M_n = \sup_i | a_{ni} |$, then

$$\sum_i | a_{ni} |^p = M_n{}^p \sum_i \left| \frac{a_{ni}}{M_n} \right| \leqslant M_n{}^p \sum_i \frac{| a_{ni} |}{M_n} \leqslant M_n^{p-1} m,$$

which gives the desired result.

If f is a general member of L_p and $\epsilon > 0$, then we can find $f_0 \in L_p(B, m)$, $f_k \in L_p(D_k, m)$, $k = 1,..., K$ so that $\| f - \sum_{k=0}^{K} f_k \| < \epsilon$. Since $\sum_i a_{ni} Q^i \sum_{k=0}^{K} f_k$ converges strongly to zero, this shows that $\sum_i a_{ni} Q^i f$ also converges strongly to zero and completes the proof. □

Finally we prove the following result which shows that for our purposes Borel spaces are enough.

THEOREM 5.4. *Let (X, Σ, m) be a finite measure space and let T be a bounded linear operator on $L_p(X, \Sigma, m)$. Given countably many functions $f_1, f_2,...$ in $L_p(X, \Sigma, m)$, there exists a Borel space (J, β, μ) so that $L_p(J, \beta, \mu)$ is isomorphic to a subspace of $L_p(X, \Sigma, m)$ and this subspace is invariant under T and contains $f_1, f_2,...$. Furthermore, this isomorphism preserves the positivity.*

Before the proof we note how to apply this theorem to our case. We start with a positive contraction T on $L_p(F, \Sigma, m)$ and assume that there is a function $h' = h1_F$ in L_p, $h' > 0$ a.e. and $\| Th' \| = \| h' \|$. If T^n converges weakly to zero, we would like to show that $\sum a_{ni} T^i f$ converges to zero in norm for each $f \in L_p$. Therefore, given a fixed f in L_p, we apply Theorem 5.4 to get an invariant subspace of L_p containing h' and f and being isomorphic to the L_p space of a Borel measure space. Then the dilation theorem applies and we proceed as before.

Proof of Theorem 5.4. Let us call a σ-algebra separable if it can be generated by countably many sets. Then we note that countably many functions on a space always generate a separable σ-algebra. In fact, if $g_n: X \to \mathbf{R}$, $n = 1, 2,...$, then they define a mapping $\psi: X \to \mathbf{R}^\infty$ as $\psi(x) = (g_1(x), g_2(x),...)$, and the σ-algebra generated by $(g_1, g_2,...)$ is just $\psi^{-1}\beta^\infty$. Here, of course, $(\mathbf{R}^\infty, \beta^\infty)$ is the cartesian product of countably many copies of the real line $\mathbf{R}$, together with the usual Borel σ-algebra. Since β^∞ is separable, we see that $\psi^{-1}\beta^\infty$ is also separable. Also note that the σ-algebra generated by countably many separable σ-algebras is itself separable. Now let $\mathscr{F}_0$ be the σ-algebra generated by $(f_1, f_2,...)$. We define a sequence of separable σ-algebras $(\mathscr{F}_0, \mathscr{F}_1,...)$ as follows. If $\mathscr{F}_n$ is defined and if it is generated by a sequence of sets $(F_{n1}, F_{n2},...)$, then $\mathscr{F}_{n+1}$ is the σ-algebra generated by the countably many functions $(T1_{F_{n1}}, T1_{F_{n2}},...)$. Also, let $\mathscr{F} \subset \Sigma$ be the σ-algebra generated by $(\mathscr{F}_0, \mathscr{F}_1,...)$. Then it is clear that the subspace of $L_p(X, \Sigma, m)$ consisting of $\mathscr{F}$-measurable L_p-functions is invariant under T and contains $(f_1, f_2,...)$. If $(g_1, g_2,...)$ is a sequence of functions generating $\mathscr{F}$ and if $\psi: X \to \mathbf{R}^\infty$ is the mapping defined as $\psi(x) = (g_1(x), g_2(x),...)$, then ψ is, of course, Σ-measurable and transports m to a measure μ on $(\mathbf{R}^\infty, \beta^\infty)$. Then it is clear that ψ also defines a positivity preserving isomorphism between $L_p(X, \mathscr{F}, m)$ and $L_p(\mathbf{R}^\infty, \beta^\infty, \mu)$. Since $(\mathbf{R}^\infty, \beta^\infty, \mu)$ is a Borel space this completes the proof. □

Remark. If in Theorem 5.1 one assumes that (a_{ni}) satisfies only (2.6) instead of (1.1), then weak-lim $T^n = \bar{T}$ only implies

$$\sum_i a_{ni} T^i (I - \bar{T}) \to 0.$$

The proof of this is the same as the proof of Theorem 5.1.

References

1. M. A. Akcoglu, Sub σ-algebras of Lebesgue spaces, to appear.
2. M. A. Akcoglu, A pointwise ergodic theorem in L_p-spaces, *Canad. J. Math.* to appear.
3. M. A. Akcoglu, Positive contractions of L_1-spaces, *Math. Z.*, to appear.
4. M. A. Akcoglu and R. V. Chacon, A Convexity Theorem for Positive Operators, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **3** (1965), 328–332.
5. M. A. Akcoglu and L. Sucheston, On operator convergence in Hilbert space and in Lebesgue space, *Periodica Math. Hungarica* **2** (1972), 235–244.
6. N. Dunford and J. T. Schwartz, "Linear Operators," Part I, Interscience, New York, 1958.
7. H. Fong and L. Sucheston, On a mixing property of operators in L_p spaces, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **28** (1974), 165–171.
8. M. Loève, "Probability Theory," Third ed., D. Van Nostrand, Princeton, NJ, 1963.
9. G. G. Lorentz, A Contribution to the Theory of Divergent Sequences, *Acta Math.* **80** (1948), 167–190.
10. G. G. Lorentz, Direct Theorems on Methods of Summability, *Canad. J. Math.* **1** (1949), 305–319.
11. H. L. Royden, "Real Analysis," Second ed., Macmillan, New York, 1968.
12. V. A. Rohlin, On the Fundamental Ideas of Measure Theory, *Mat. Sb.* **25** (1949), 107–150.
13. R. Sato, On Akcoglu and Sucheston's Operator Convergence Theorem in Lebesgue Space, *Proc. Amer. Math. Soc.* **40** (1973), 513–516.